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Quaternion Fourier Transform for Colour Images

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Abstract— **The Fourier transforms plays a critical role in broad range of image processing applications, including enhancement, restoration, analysis and compression. For filtering of gray scale images 2D Fourier transform is an important tool which converts the image from spatial domain to frequency domain and then by applying filtering mask filtering is done. To filter color images, a new approach is implemented which uses hyper complex numbers (called as Quaternion) to represent color images and uses Quaternion Fourier transform for filtering. This transform allows color images to be transformed as whole, rather than as color separated components. This paper is concerned with frequency domain noise reduction of colour images using Quaternion Fourier transform. The approach is based on obtaining quaternion Fourier transform of colour image and applying the Gaussian filter to it in the frequency domain. The filtered image is then obtained by calculating the inverse quaternion fourier transform, which explains the accuracy of this method.**

Keywords— **Hypercomplex numbers, Quaternion Fourier Transform(QFT), Gaussian LPF and HPF.**

I. INTRODUCTION

Fourier transform have been widely used in signal and image processing ever since the discovery of Fast Fourier transform in 1965 which made the computation of discrete Fourier transform feasible using a computer[1]. Fourier transform finds major application in image processing, such as image filtering, image analysis, image reconstruction and image compression. In general, there are two main approaches for filtering an image. The first is the convolution of an image and kernel in the spatial domain. The second is the multiplication of an image's Fourier transform with a filter in the frequency domain.

Until recently, there was no definition of Fourier transform applicable to colour images in holistic manner. The idea of computing the Fourier transform of a colour images has only recently been realized. It is possible to separate a colour image into three scalar images and compute the Fourier transforms of these images separately but this method is inefficient. In this paper we are concerned with the computation of a single, holistic, Fourier transform which treats a colour image as a vector field and application of this transform to colour image filtering such as low pass filtering for smoothing and high pass filtering for sharpening. The application of a quaternion Fourier transform to colour images is based on representing colour image pixels using quaternion discovered by Hamilton in 1843 [2]. The first definition of a quaternion Fourier transform was that of Ell [3],[4],[5] and the first application of a quaternion Fourier transform of colour images was reported in 1996 [4] using a discrete version of Ell' transform.

II. QUATERNION

The concept of the quaternion was introduced by Sir. William Hamilton in 1843 [2]. It is the generalization of a complex number. A complex number has two components: the real and the imaginary part. However, the quaternion has four components i.e. one real part and three imaginary parts and can be represented in Cartesian form as:

$$q = w + xi + yj + zk \qquad (1)$$

Where w, x, y, and z are real numbers and i, j and k are complex operators which obey the following rules;

$$\begin{gathered} i * j = k, \quad j * k = i, \quad k * i = j \\ j * i = -k, \quad k * j = -i, \quad i * k = -j \qquad (2) \\ i^2 = j^2 = k^2 = i * j * k = -1 \end{gathered}$$

From these rules, it is clear that multiplication is not commutative. The quaternion conjugate is,

$$\overline{q} = w - xi - yj - zk \qquad (3)$$

and the modulus of a quaternion is given by,

$$|q| = \sqrt{w^2 + x^2 + y^2 + z^2} \qquad (4)$$

A quaternion with zero real part is called a pure quaternion and a quaternion with unit modulus is called a unit quaternion.

$$q = \frac{i + j + k}{\sqrt{3}} \qquad (5)$$

The imaginary part of a quaternion has three components and may be associated with a 3-space vector. For this reason, it is sometimes useful to consider the quaternion as composed of a vector part and a scalar part .Thus q can be expressed as;

$$q = S(q) + V(q) \qquad (6)$$

where *S(q)* is the real or scalar part i.e *S(q)* = *w* and *V(q)* is the vector part which is a composition of three imaginary components;

$$V(q) = xi + yj + zk$$

A. *Representation of colour pixel as Quaternion*

Colour image pixel has three components viz, Red, Green and Blue, and they can be represented in quaternion form using pure quaternion [6]. For images in RGB colour space, the three imaginary parts of a pure quaternion can be used to represent the red, green and blue colour components. A pixel at image coordinates (x, y) in an RGB image can be represented as,

$$f(x,y) = r(x,y)i + g(x,y)j + b(x,y)k \tag{7}$$

where *r(x, y)*, *g(x, y)* and *b(x, y)* are red, green and blue components of a colour image pixel respectively.

Using quaternion to represent the RGB colour space, the three colour channels are processed equally in operations such as multiplication. The advantage of using quaternion based operations to manipulate colour information in an image is that we do not have to process each colour channel independently, but rather, treat each colour triple as a whole unit. We believe, by using quaternion operations, higher colour information accuracy can be achieved because a colour is treated as an entity.

III. QUATERNION FOURIER TRANSFORM

Based on the concept of quaternion multiplication and exponential, the Discrete Quaternion Fourier Transform (DQFT) has been introduced[7]. Due to the non-commutative property of the quaternion, there are three different types of DQFT defined: the left side DQFT, the right side DQFT and the two sides DQFT represented as,

Two-sided DQFT (Type-1)

$$F_{L-R}(u,v) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} e^{-\mu 2\pi \frac{xu}{M}} f(x,y) e^{-\mu 2\pi \frac{vy}{N}} \tag{8}$$

Left-sided DQFT (Type-2)

$$F_L(u,v) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} e^{-\mu 2\pi \left(\frac{xu}{M} + \frac{yv}{N} \right)} f(x,y) \tag{9}$$

Right-sided DQFT (Type-3)

$$F_R(u,v) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-\mu 2\pi \left(\frac{xu}{M} + \frac{yv}{N} \right)} \tag{10}$$

Similarly, the Inverse Discrete Quaternion Fourier Transforms (IDQFT) can be defined for the three types of QFT respectively as,

Left-sided IDQFT:

$$f(x,y) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} e^{\mu 2\pi \left(\frac{xu}{M} + \frac{yv}{N} \right)} F_L(u,v) \tag{11}$$

Right-sided IDQFT:

$$f(x,y) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} F_R(u,v) e^{\mu 2\pi \left(\frac{xu}{M} + \frac{yv}{N} \right)} \tag{12}$$

Two-sided IDQFT:

$$f(x,y) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} e^{\mu 2\pi \frac{xu}{M}} F_{L-R}(u,v) e^{\mu 2\pi \frac{vy}{N}} \tag{13}$$

μ is any unit pure quaternion. μ determines a direction in color space and an obvious choice for color images is the direction corresponding to the luminance axis which connects all the points r=g=b. In RGB color space this is the gray line.

IV. QUATERNION FILTERING

The frequency domain is the space defined by values of the Fourier Transform and its frequency variables (u, v). In this section, we define different types of quaternion smoothing and sharpening filters. In the case where $h(x, y)$ (impulse response of quaternion filter) has the even symmetry relation,

$$h(x,y) = h(-x,-y) \tag{14}$$

QFT of $h(x, y)$ denoted by $H_q(u,v)$ also has the same symmetry relation and the relation between the quaternion convolution and the QFT can be simplified to,

$$G(u,v) = H_q(u,v) F(u,v) \tag{15}$$

the quaternion convolution operation in the spatial domain corresponds to the product operation in the frequency domain [8]. This is the same as the case of the conventional convolution. where $F(u,v)$ is the quaternion Fourier transform of the color image to be filtered and $G(u,v)$ is the quaternion Fourier transform of the filtered output image. The objective is to select a quaternion filter transfer function $H_q(u,v)$ that yields $G(u,v)$ The filtered image is obtained simply by taking the inverse quaternion Fourier transform of $G(u,v)$.

A. *Quaternion Low pass filtering*

The edges and other sharp transitions in the pixels of an image contribute significantly to the high frequency content of its Fourier transform [6]. Hence, smoothing of a colour image is achieved in the frequency domain by attenuating the specified range of high frequency components in the quaternion Fourier transform of the image. Now we propose two types of quaternion low pass filters whose impulse responses satisfy Eq. (14): ideal and Gaussian [9].

1) *Ideal quaternion Low pass filters*: The simplest low pass quaternion filter is a filter that cuts off all high frequency components of the quaternion Fourier Transform that are at a distance greater that a specified distance D_0 from the origin of the transform. Such a filter has the transfer function,

$$H_q(u,v) = \begin{cases} 1, & D(u,v) \le D_0 \\ 0, & D(u,v) \ge D_0 \end{cases} \qquad (16)$$

where D_0 is a non-negative quantity, and $D(u,v)$ is the distance from point (u,v) to the origin of the frequency rectangle $(M/2, N/2)$, given by

$$D(u,v) = \left[\left(u - \frac{M}{2} \right)^2 + \left(v - \frac{N}{2} \right)^2 \right]^{\frac{1}{2}} \qquad (17)$$

Fig.1 shows the results of ideal low pass filtering in the hyper complex spectral domain. In the first column of Fig.1, are the original image and its spectral modulus. The second, third and fourth columns, in the same Fig. , show the results of ideal low pass filtering the image by masks with radii 10, 20 and 30. As expected the resulting images are blurred consistent with this operation. As the filter radius increases, less and less power is removed, resulting in less blurring.

2) *Gaussian Quaternion low pass filters:* The two dimensional Gaussian low pass filter transfer function with the cut-off frequency at a distance D_0 is given by,

$$H_q(u,v) = e^{-\frac{D^2(u,v)}{2\sigma^2}} \qquad (18)$$

where σ is a measure of Gaussian spread and is equal to D_0 and $D(u,v)$ as in Eq.(17) is the distance from the origin of the Fourier transform. Fig.2 shows the results of Gaussian quaternion low pass filtering the Lena image in the hyper complex spectral domain. As in the case of the Ideal low pass filter, we note a smooth transition in blurring as a function of increasing cutoff frequency. The Gaussian low pass filter did not achieve as much smoothing as the Butterworth low pass filter of order 2 for the same value of cutoff frequency.

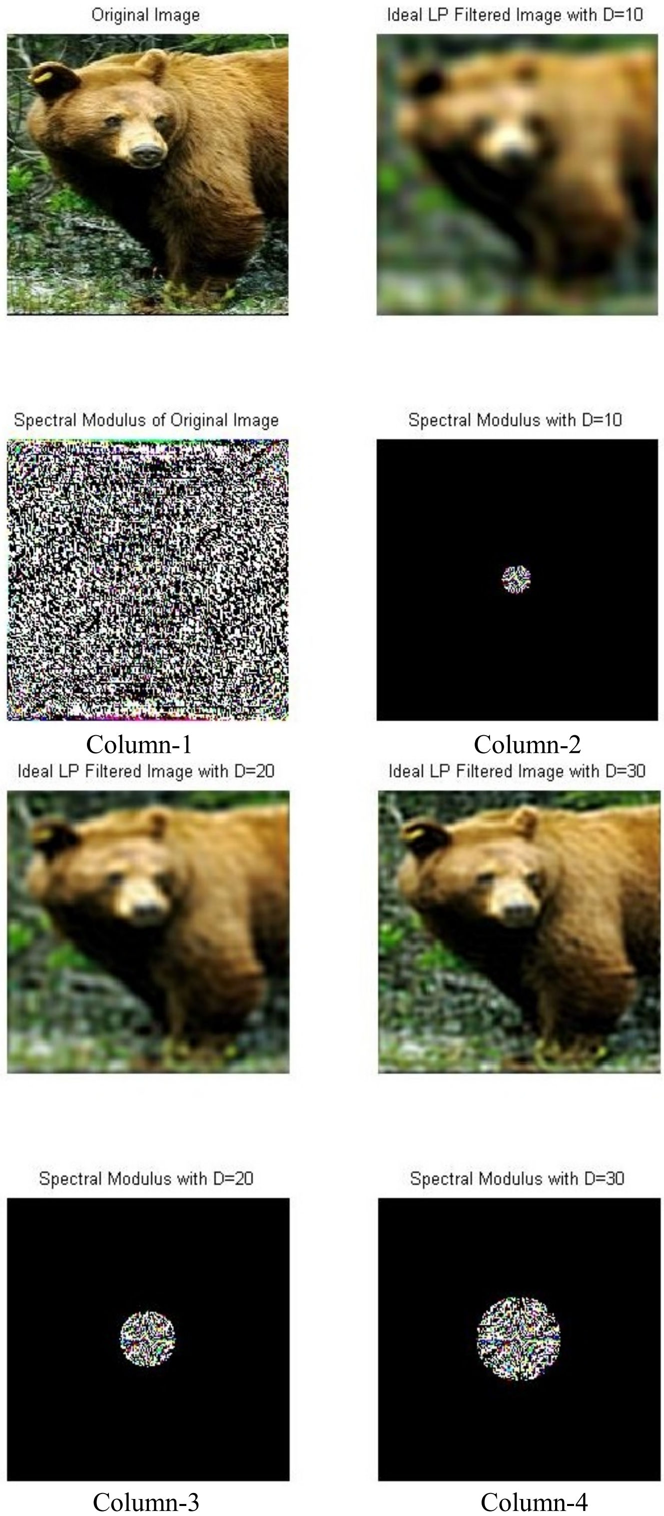

Fig. 1: The results of ideal low pass filtering based on type 2 QFT, first column shows the original noisy colour image and its spectral modulus. The low pass filtering of the image by the masks with radii 10,20 and 30 and the corresponding type two inverse QFT(Left-Side) images are likewise shown in the second ,third and fourth columns, respectively.

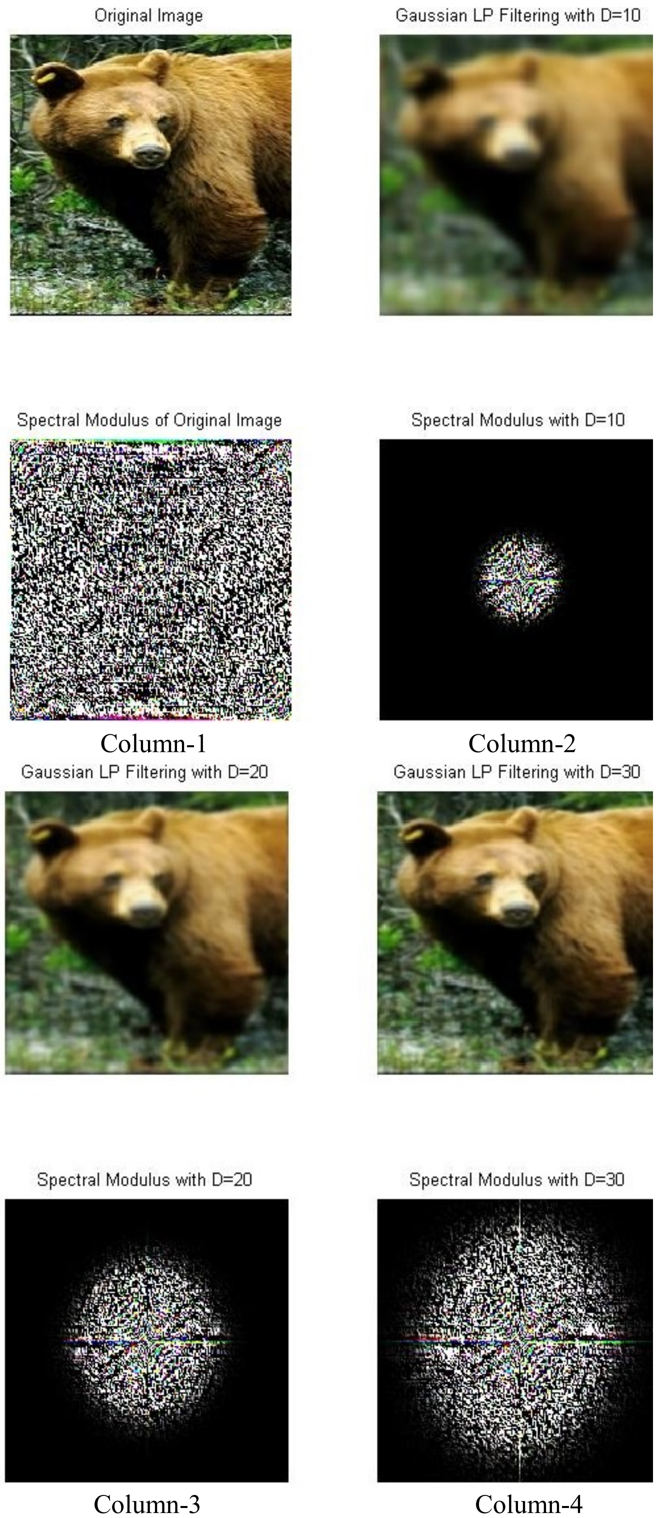

Fig. 2: The results of Gaussian low pass filtering based on type 2 QFT, first column shows the original noisy colour image and its spectrum modulus. The low pass filtering of the image by the masks with radii 10,20 and 30 and the corresponding type two inverse QFT(Left-Side) images are likewise shown in the second ,third and fourth columns, respectively.

B. Quaternion High pass filtering

The edges and other abrupt changes in pixels are associated with high frequency components; image sharpening can be achieved in the frequency domain by a high pass filtering process, which attenuates the low frequency components without disturbing high frequency information in the quaternion Fourier transform. Because the intended function of the quaternion high pass filter is to perform the reverse operation of the quaternion low pass filter, the transfer function of the high pass filters can be obtained using the relation:

$$H_{hpf}(u,v) = 1 - H_{lpf}(u,v) \qquad (19)$$

Where $H_{lpf}(u,v)$ and $H_{hpf}(u,v)$ is the transfer function of the corresponding low pass quaternion filter. In this section also, we propose two types of quaternion high pass filters that satisfy Eq. (14): ideal and Gaussian [9].

1) *Ideal Quaternion High pass filters:* A two dimensional quaternion high pass filter is defined as,

$$H_q(u,v) = \begin{cases} 0, & D(u,v) \le D_0 \\ 1, & D(u,v) \ge D_0 \end{cases} \qquad (20)$$

Where D_0 is a non-negative quantity, and $D(u,v)$ is the distance from point (u,v) to the origin of the frequency rectangle, as given by Eq(17).

Fig.3 shows the results of ideal high pass filtering in the hyper complex spectral domain. In the first column, are the original image and its spectral modulus. The second, third and fourth columns, in the same Fig. show the results of high pass filtering the image by masks with radii 10, 20 and 30. Again, as expected, the image contains content where there is rapid luminance and chrominance variation. From this Fig. It can be seen that the sharpening of the image increases as radius of mask increases.

2) *Gaussian Quaternion High pass filters:* The transfer function of the Gaussian high pass quaternion filter with cut-off frequency locus at a distance D_0 from the origin is given by,

$$H_q(u,v) = 1 - e^{-\frac{D^2(u,v)}{2\sigma^2}} \qquad (21)$$

where σ is a measure of Gaussian spread and is equal to D_0 and $D(u,v)$ as in Eq.(17)

Fig.4 shows the results of Gaussian high pass filtering the image in the hyper complex spectral domain. In the first column, are the original image and its spectral modulus. The second, third, and fourth columns, in the same Fig. show the results of Gaussian high pass filtering the image by masks with radii 10, 20 and 30. The results obtained with Gaussian high pass filtering are smoother than with ideal high pass filters.

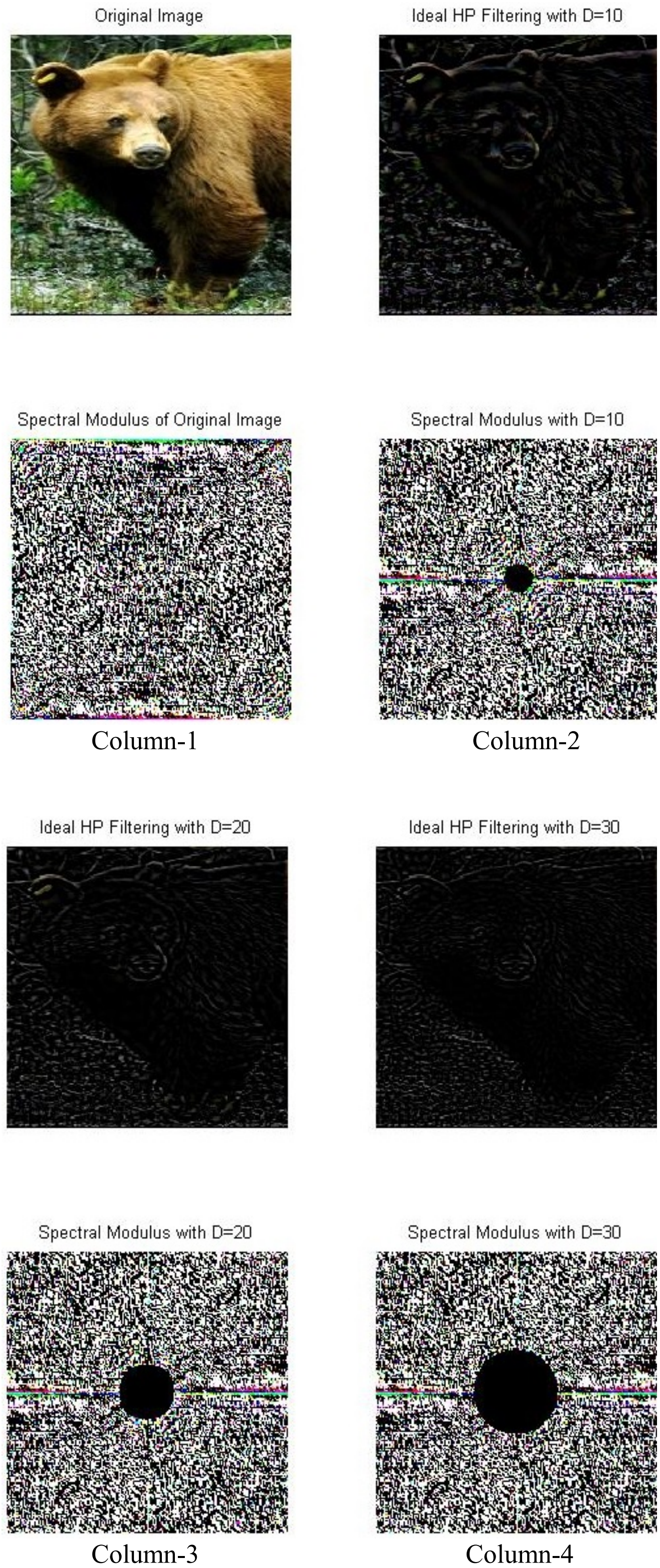

Fig. 3: The results of ideal high pass filtering based on type 2 QFT, first column shows the original colour image and its spectrum modulus. The high pass filtering of the image by the masks with radii 5,10 and 20 and the corresponding type two inverse QFT(Left-Side) images are likewise shown in the second ,third and fourth columns, respectively.

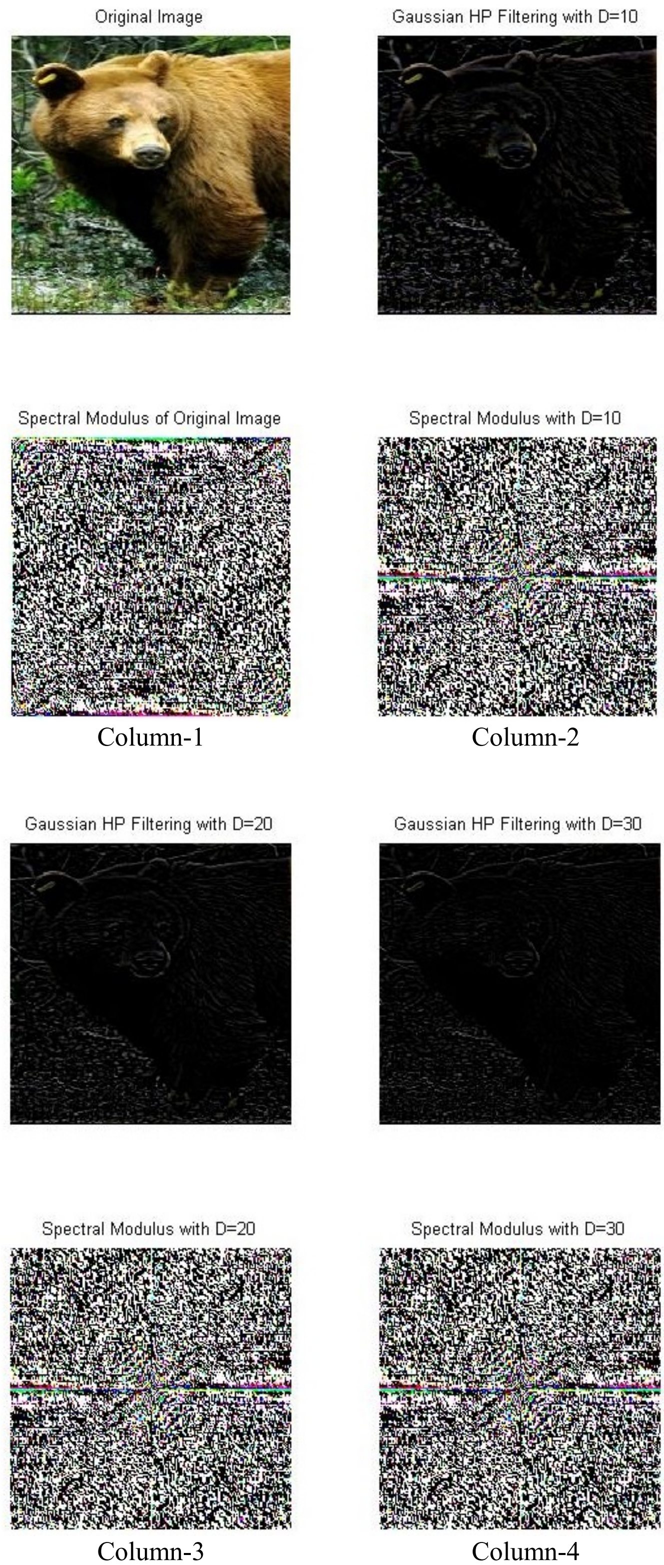

Fig. 4: The results of Gaussian high pass filtering based on type 2 QFT, first column shows the original colour image and its spectrum modulus. The high pass filtering of the image by the masks with radii 5,10 and 20 and the corresponding type two inverse QFT(Left-Side) images are likewise shown in the second ,third and fourth columns, respectively.

V. CONCLUSIONS

This work demonstrates that the Quaternion Fourier Transform is well suited for describing the spectral content of colour images. Similar to gray scale images, colour images represented as quaternion valued images can also be transformed into the Frequency domain and can be represented as quaternion frequency signals, based on which different image processing techniques such as filtering can be performed efficiently. Filtering in quaternion frequency domain has the advantage that the colour triples are processed as a whole unit rather than dealing with RGB channels separately. We believe more accurate colour information can be preserved this way, since all colour channels are processed as a single unit. And we would like to point out that, the Quaternion Fourier transform is not limited to this, but also can be applied to other colour image processing fields, such as image registration, edge detection, and data compression.

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